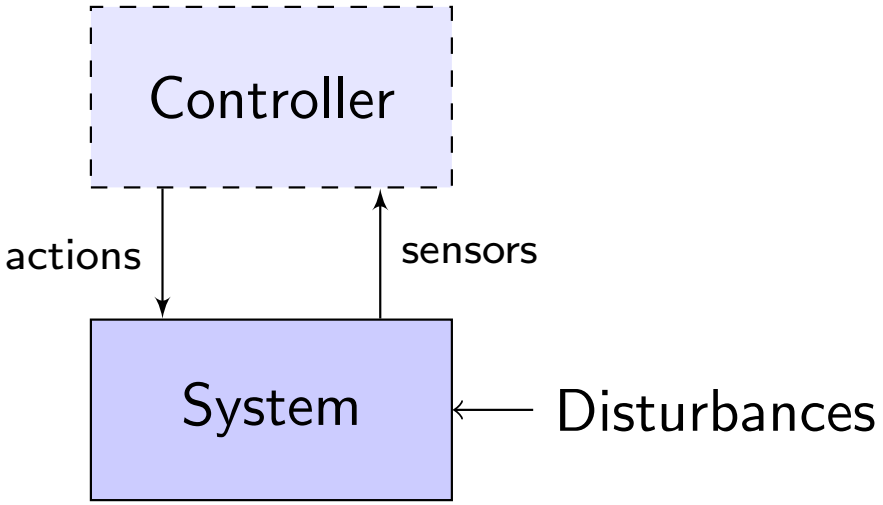
Controller Synthesis for Timed Systems

Ocan Sankur

Controller Synthesis



Goal: Given a system (model), automatically compute a controller to ensure the system always has desired behavior.

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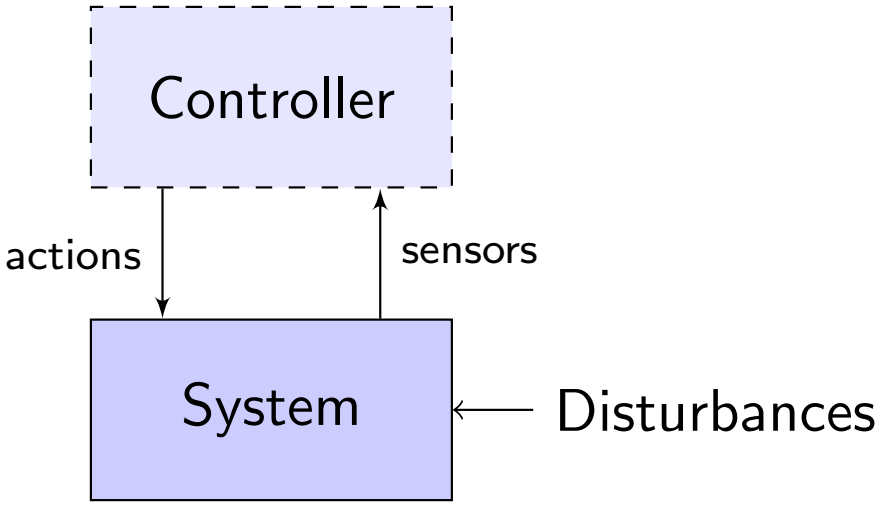

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Timed Systems: **Discrete** systems with timing constraints / objectives:

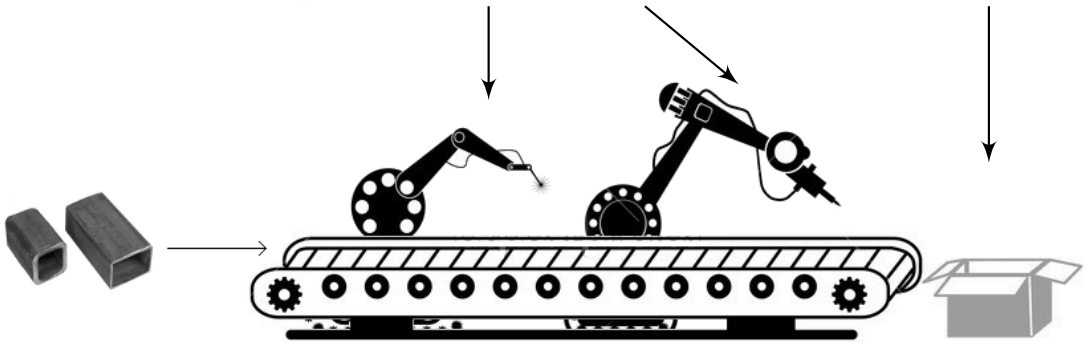
1. The system evolves with *discrete events*
2. We make the passage of time explicit, e.g. we consider execution time, deadlines etc.

Next: example

Production Line Example

Objectives

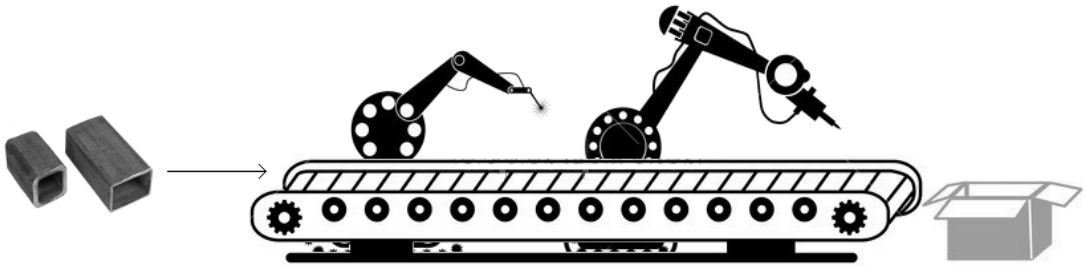
- **Small** or **Large** objects arrive every $\geq$ **10** seconds.
- Small objects must be **painted**, **drilled once**, and **finalized**.
- Large objects must be **painted** **drilled twice** **finalized**.



Constraints

- The drill must be idle for $\geq$ **3** seconds between each drill.
- No more than one object on the line at any moment.
- Painting, drilling take **1-2 seconds**, and finalizing **1 second**.

Production Line Example



This is a timed system:

- We observe **discrete events**: Object has arrived, object is small, painting starts, painting finishes, etc.
- **Timing constraints**. . .

Controller Synthesis: Find a control policy that dictates which action to take at which moment to satisfy all the constraints.

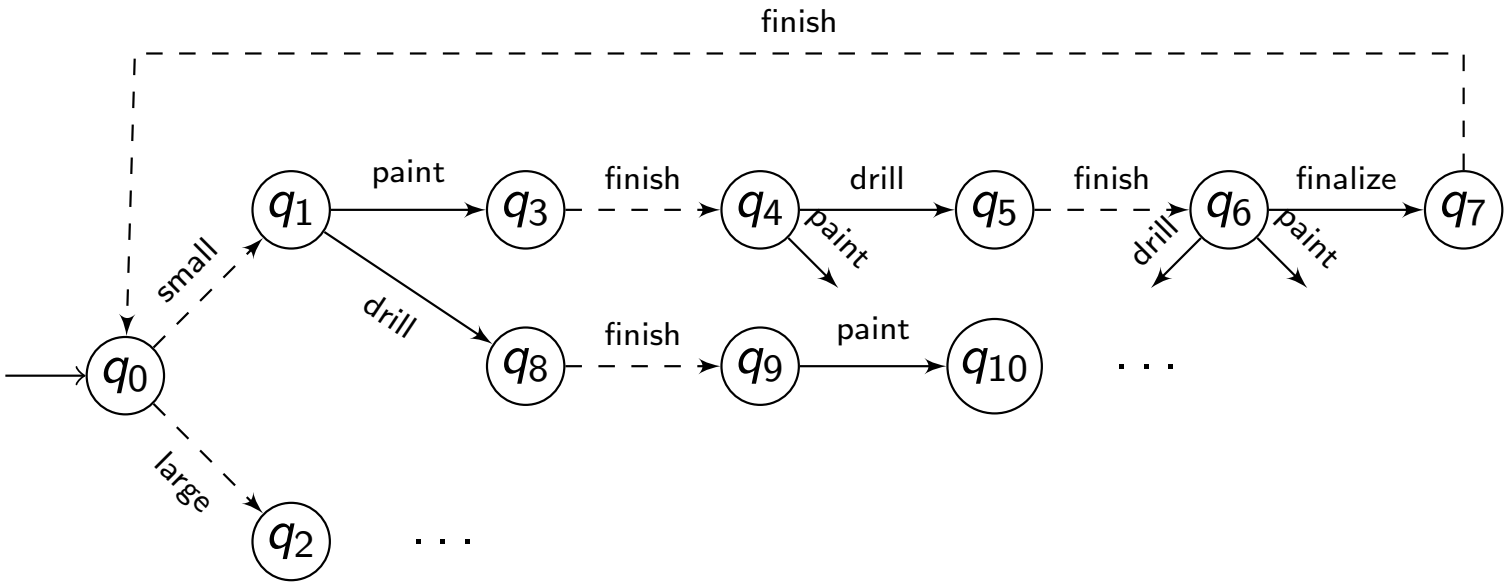
Outline

Outline

Finite Games

Formalism for **discrete control problems** (no explicit time):

- Nodes are the configurations of the system
- Plain edges: control actions
- Dashed edges: uncontrollable events (coming from the environment)



Control Strategy: A mapping from states to control actions
Control Objective: Each arriving object is processed and finalized.
Winning Strategy: All executions under the strategy satisfy the objective.

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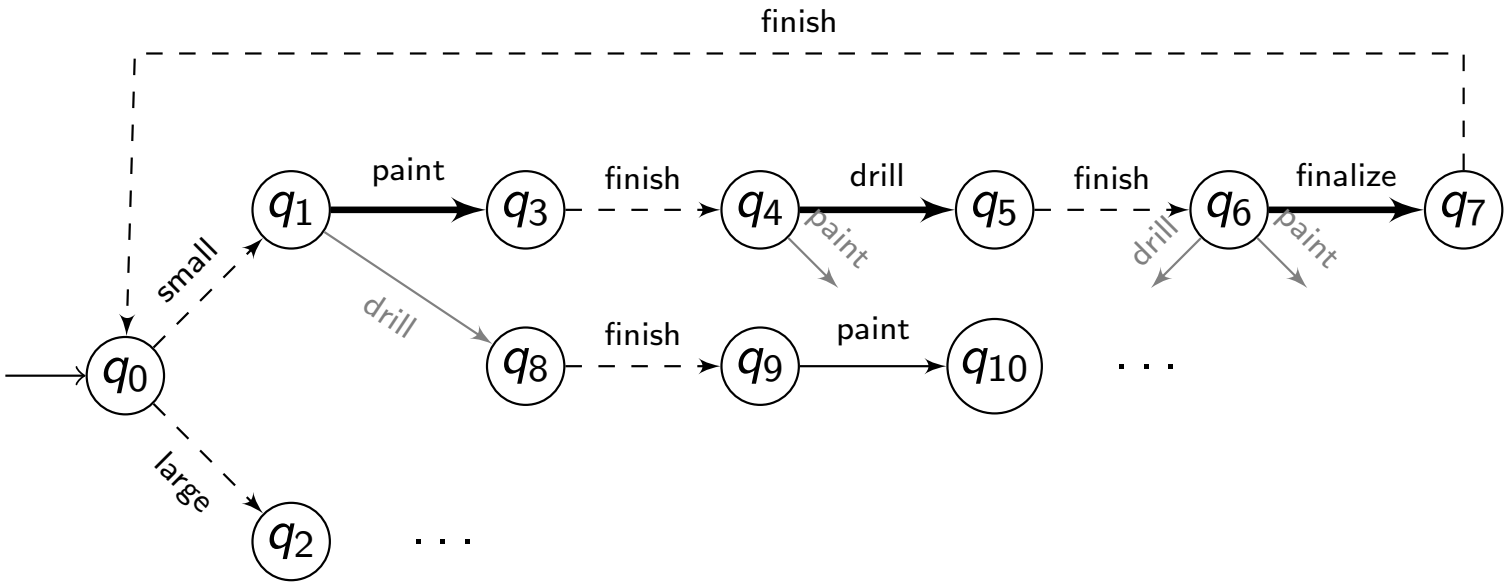

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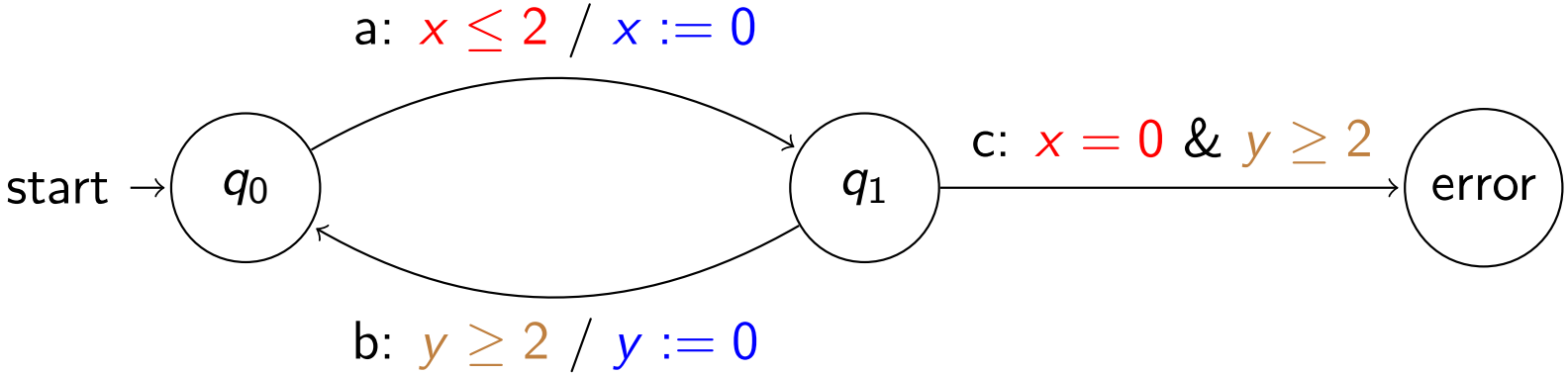
Timed Automata

Timed automata = Finite automata + Clocks. [Alur and Dill 1994]



Run of a TA

$$(q_0, (x = 0, y = 0)) \xrightarrow{1.7} (q_0, (x = 1.7, y = 1.7)) \xrightarrow{a} (q_1, (x = 0, y = 1.7))$$
$$\xrightarrow{0.5} (q_1, (x = 0.5, y = 2.2)) \xrightarrow{b} (q_0, (x = 0.5, y = 0)) \ldots$$

A timed automaton is used to describe the **set of all runs**

all possible behaviors of the system

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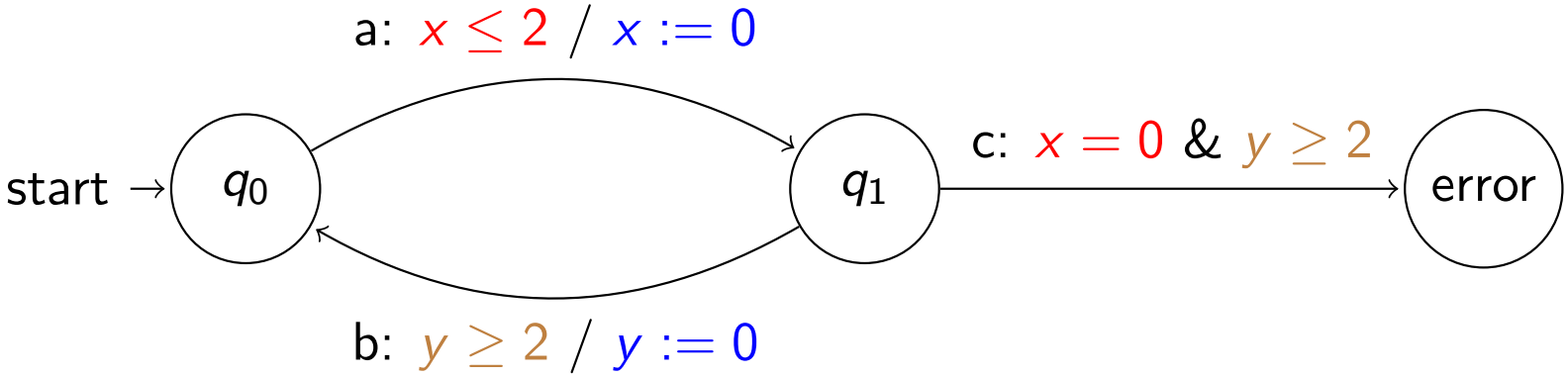

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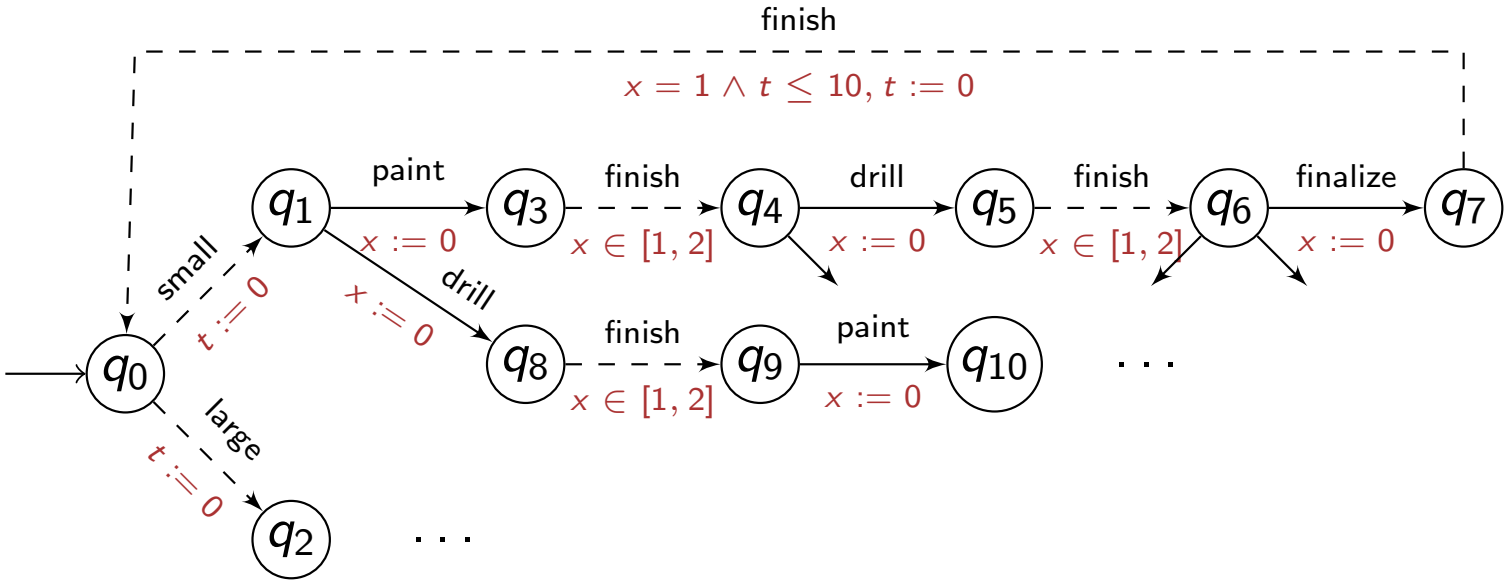
Timed Systems can be modeled by timed automata:

- a, b, c are discrete events,
- Clocks are used for timing constraints

$\rightarrow$ Let us define **games** played on timed automata

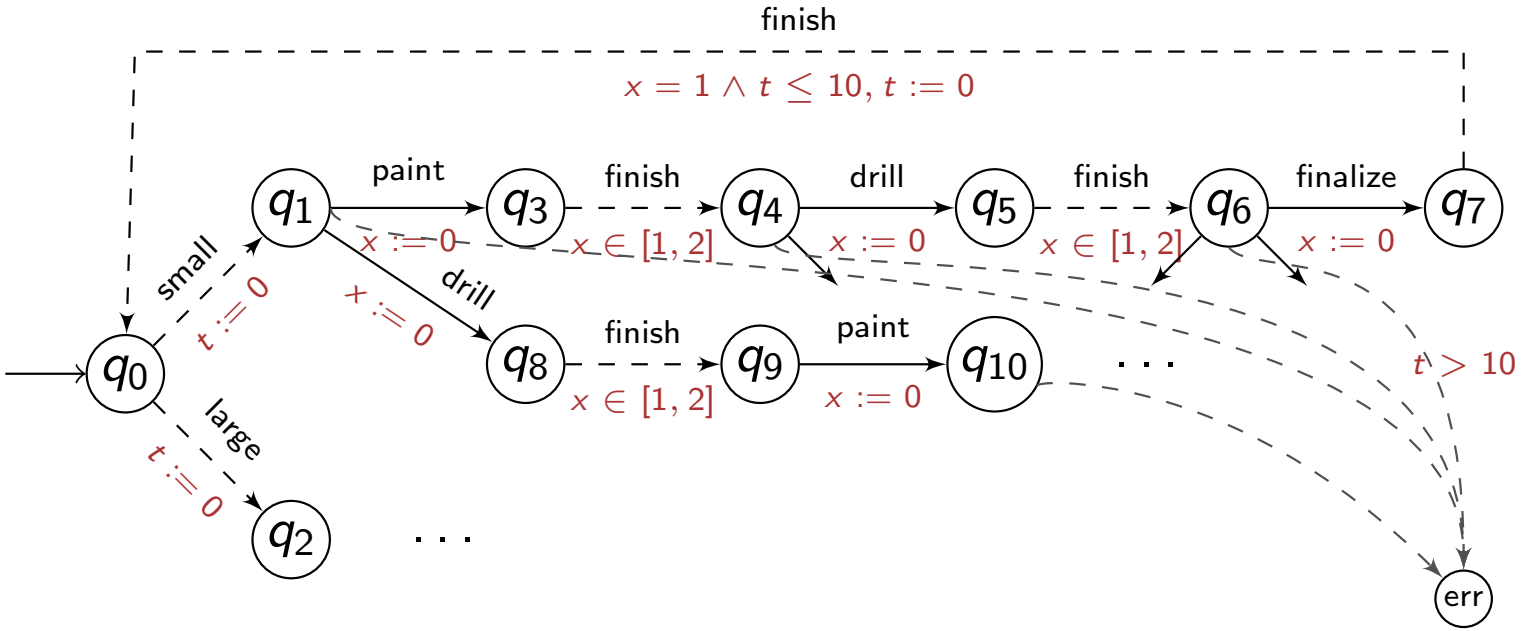
Timed Games



Control objective: Process each object within 10 seconds – against **any** environment behavior (choice of objects, painting durations, etc.)

Control Strategy: A function mapping each state to a delay and action to be taken $(\mathcal{L} \times \mathbb{R}^C \to \mathbb{R} \times A)$

Timed Games



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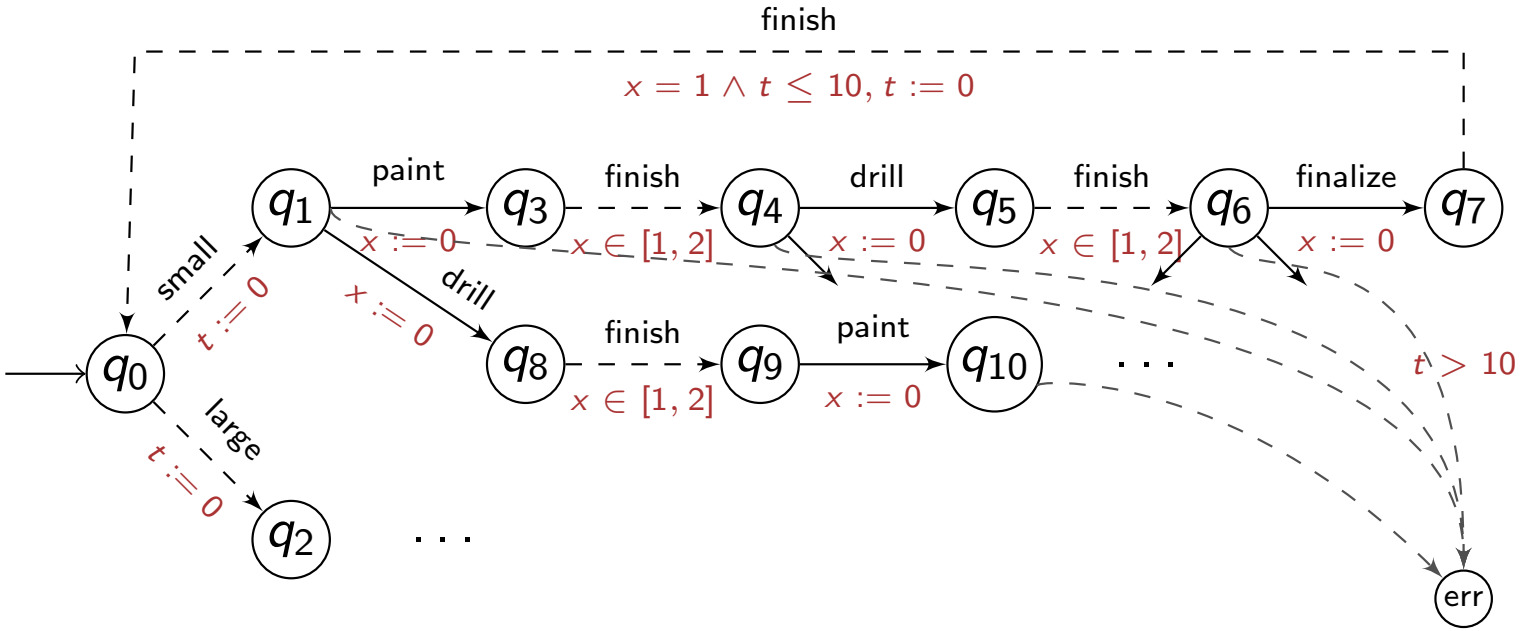

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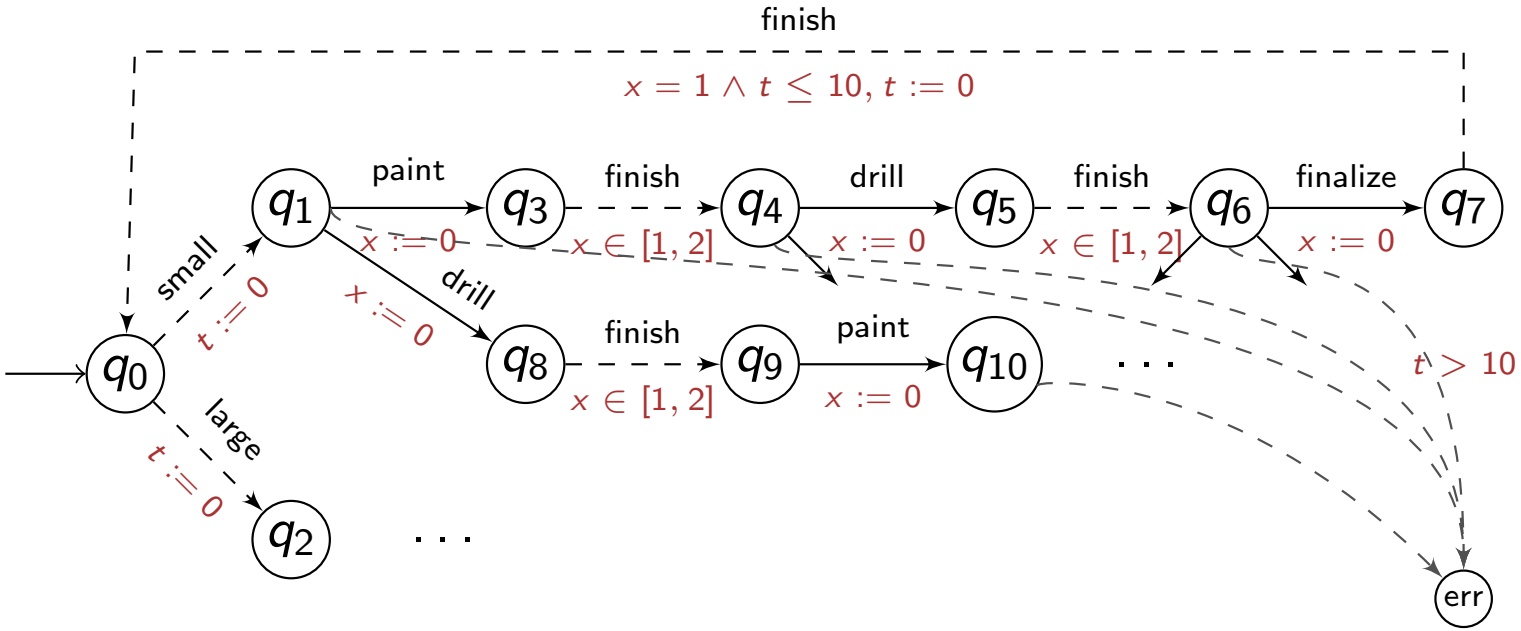

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Goal of this tutorial: Describe algorithms to solve controller synthesis using timed games

Outline

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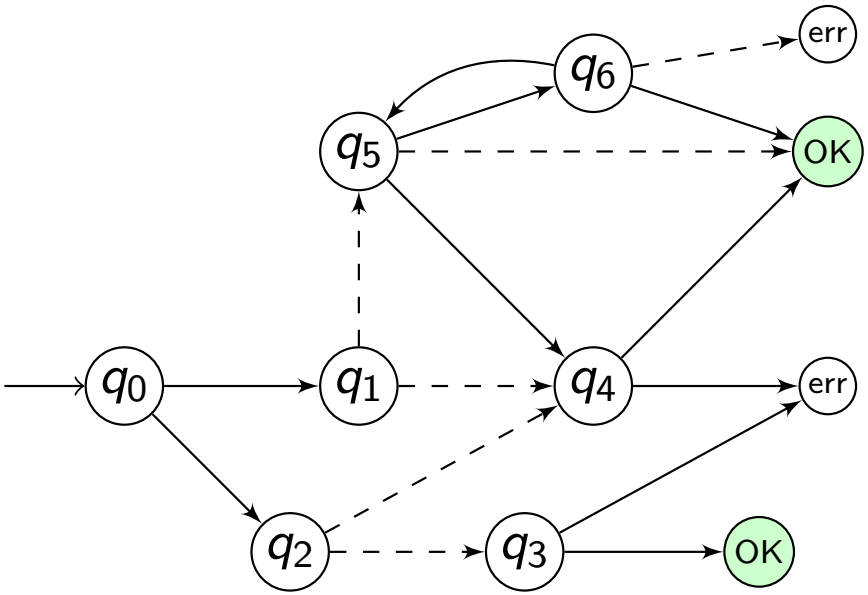
Algorithm for Finite Games

Goal: Reach the `OK` state, starting from q_0



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Step by step: which are the states from which we can win in 1 step?

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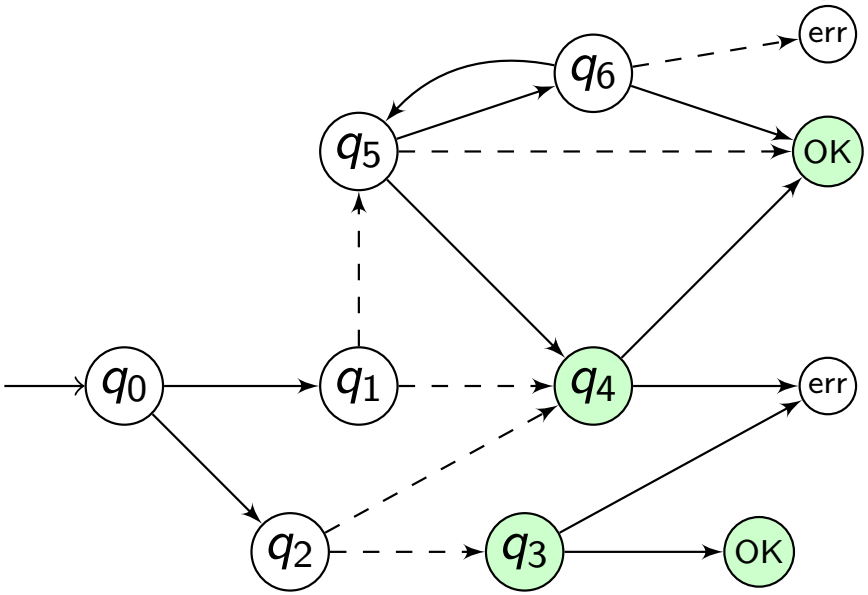

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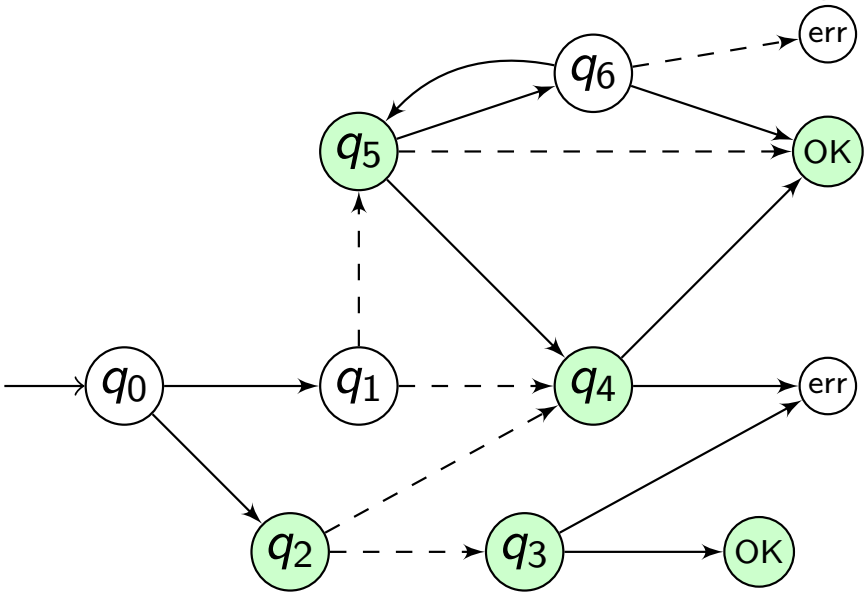

Algorithm

Step by step: which are the states from which we can win in 2 step?

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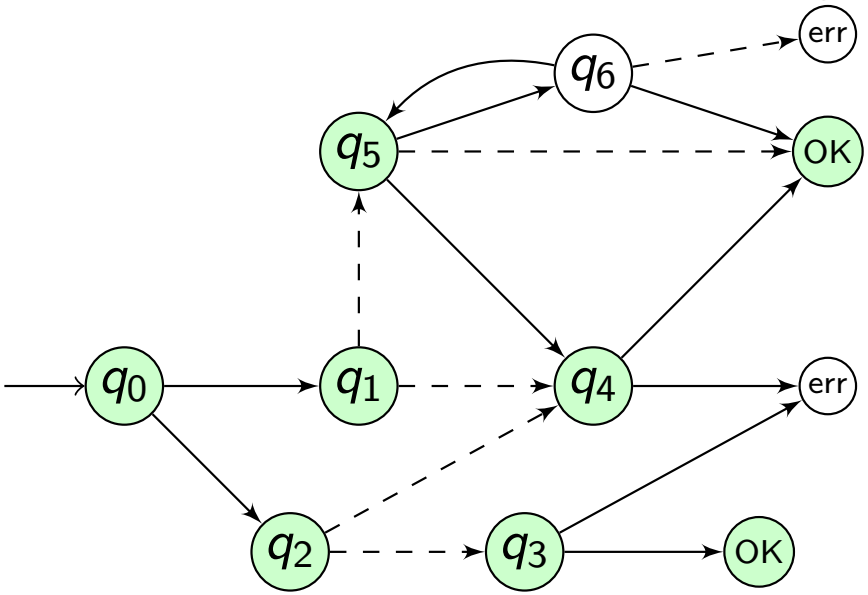

Algorithm

Step by step: which are the states from which we can win in 3 step?

Green states are **winning states**: there is a winning control strategy

Algorithm for Finite Games

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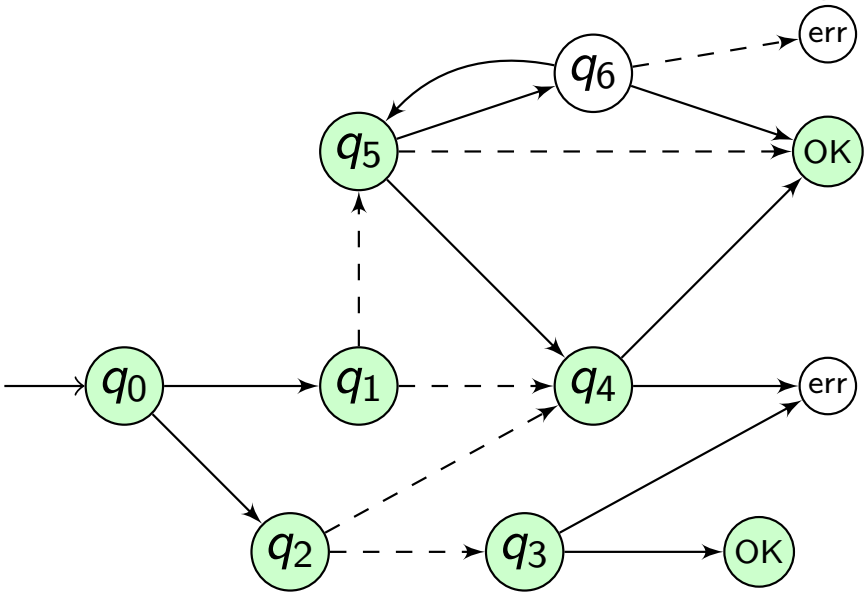


Formally, controllable predecessors of a set of states T is:

$$\mathrm{CPRE}(T) = \{q \mid \exists q' \in T, (q \to q' \vee q \dashrightarrow q') \wedge \forall q \dashrightarrow q'', q'' \in T\}.$$

The set of all winning (green) states is obtained as follows:

$X := \{\mathrm{OK}\};$
while $\mathrm{CPRE}(X) \not\subseteq X$ do
    $X := X \cup \mathrm{CPRE}(X);$

Finite Games

Fixpoint: $\mu X.(\{\mathsf{OK}\} \cup X \cup \mathsf{CPRE}(X))$.

Algorithm

The controller synthesis problem in finite games can be solved in polynomial time for finite games, by computing the fixpoint of CPRE.

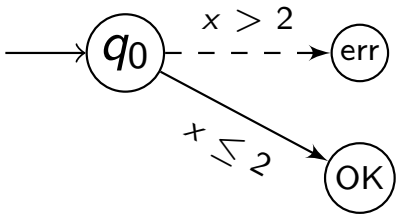
Similar results exist of ω-regular winning conditions, and temporal logics.

Memoryless strategies always exist: Control actions are a function of the current state, not the past history.

Outline

How About Timed Games?

We want to do the same for timed games.



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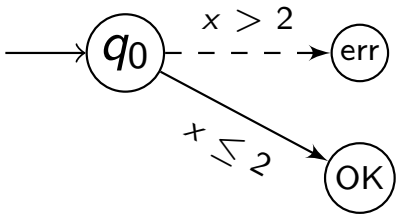

We have $\mathrm{CPRE}(\{\mathrm{OK}\}) = \{(q_0, x) \mid x \in \mathbb{R}, 0 \leq x \leq 2\}$

which is **infinite** and **uncountable**!

Enumerating successors or predecessors do not make any sense...

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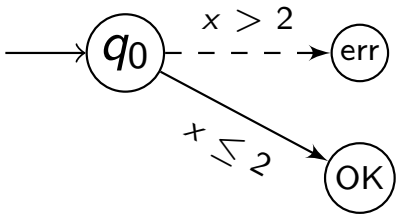

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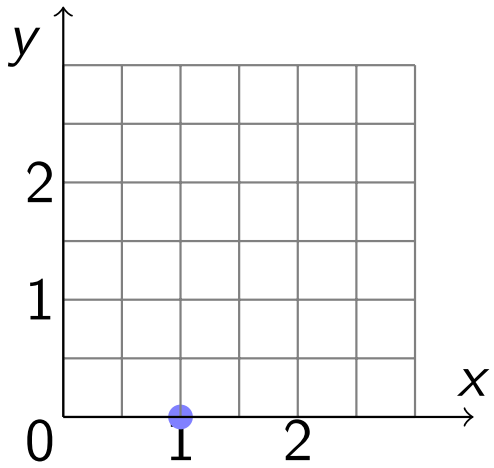
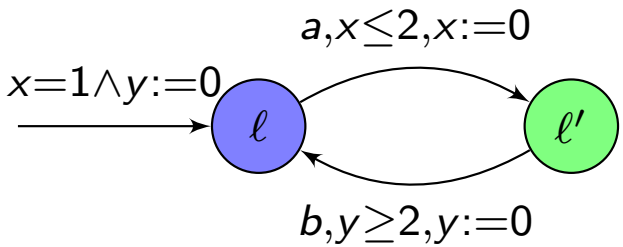
$\rightarrow$ Data structure to represent the state space of timed games

Zones - Intuition

Symbolic representation: represent **subsets** of states using **polyhedra**
States are elements of $\mathcal{L} \times \mathbb{R}^C_{\geq 0}$, where C is the set of clocks.





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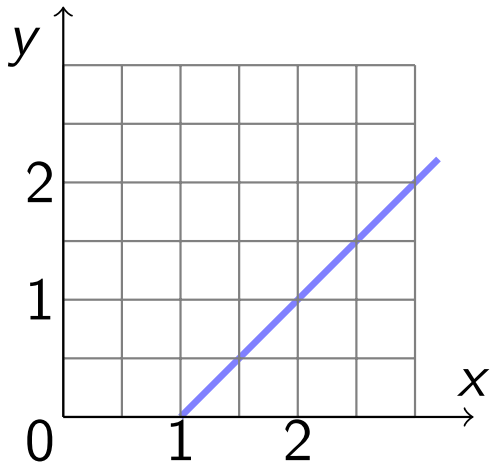

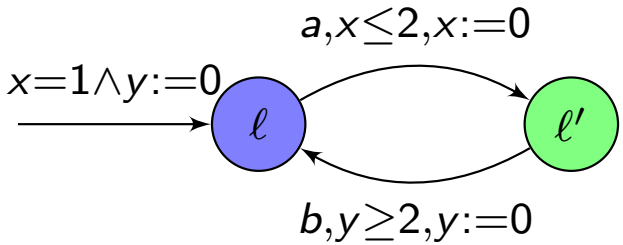

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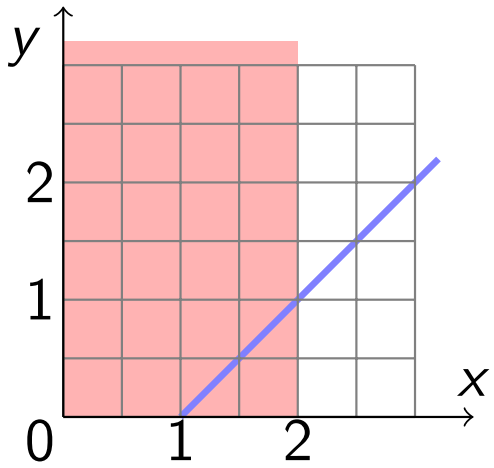

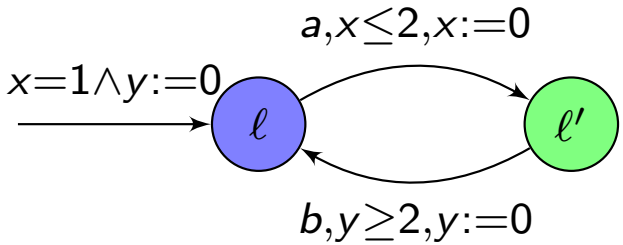

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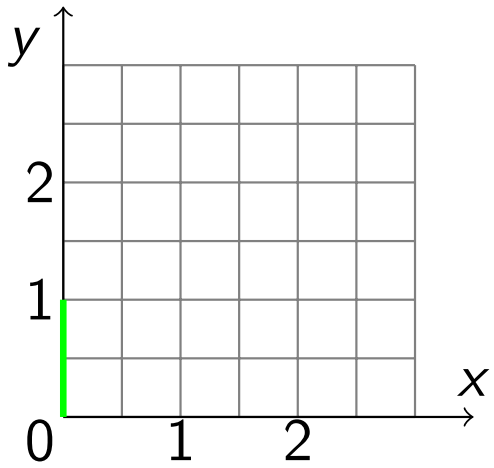

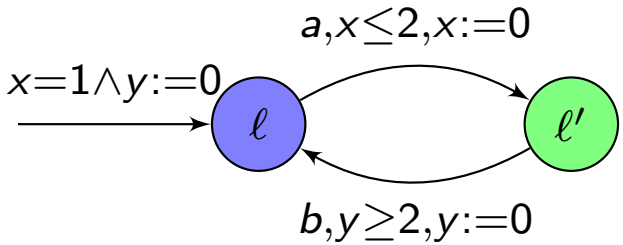

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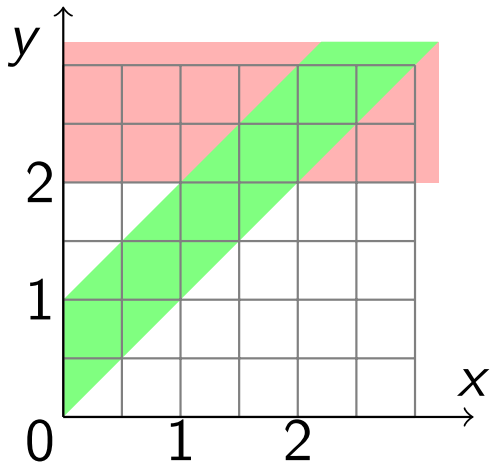

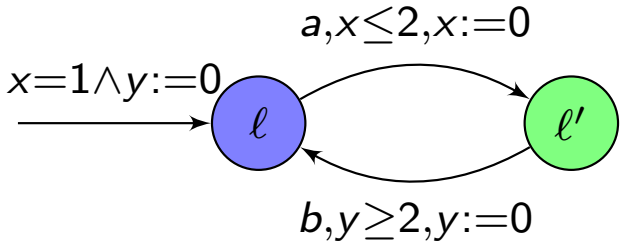

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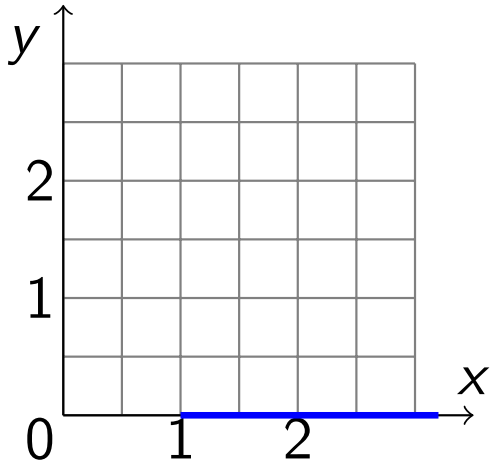

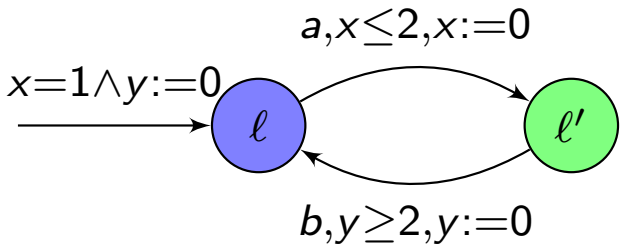

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Move back to ℓ: $\mathrm{Post}_b(\ell', 0 \leq y - x \leq 1) = (\ell, y = 0 \wedge x \geq 1)$, etc.

Timed Automata Verification

Given a sequence of edges $e_1 \dots e_n$, $\text{Post}_{e_1 \dots e_n}$ is expressible as a zone.

Using zones one can **exhaustively** explore the state space.

Zones were first used for verification by [Berthomieu, Menasche 1983]

Verification questions:

- Does the timed automaton have a run that visits an error state?
- Do all the runs of the timed automaton satisfy a given ω-regular condition (e.g. a temporal logic formula)?

Theorem

Verification of timed automata for safety properties is PSPACE-complete

Same complexity for *omega*-regular properties

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Timed games: Can we define a controllable predecessors operator?

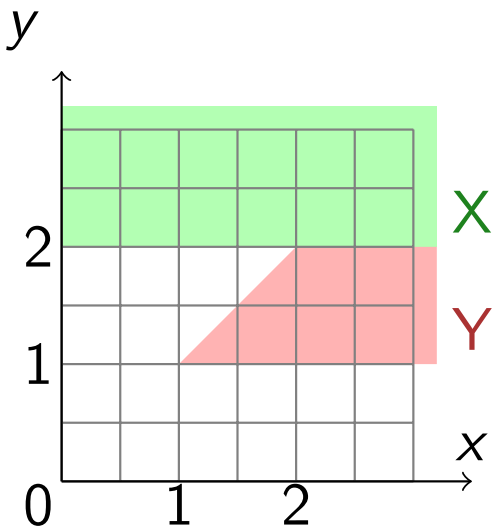
CPRE for Timed Games

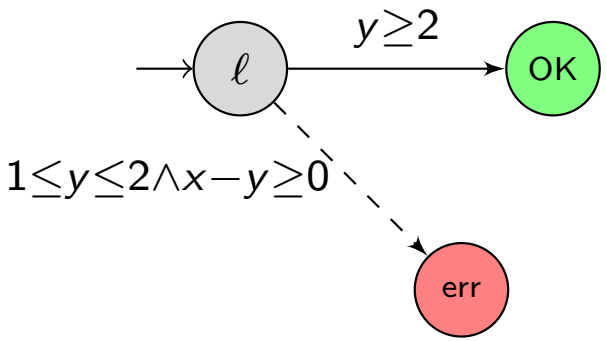
Denote $\mathcal{Q} = \mathcal{L} \times \mathbb{R}^{C}_{\geq 0}$.

Safe Time-Predecessors

Given $X, Y \subseteq \mathcal{Q}$, from which states can we delay into X while avoiding Y?

$$\mathrm{Pred}_{\geq 0}(X, Y) = \{q \in \mathcal{Q} \mid \exists d \geq 0, q + d \in X \wedge \forall d' \in [0, d], q + d' \notin Y\}.$$





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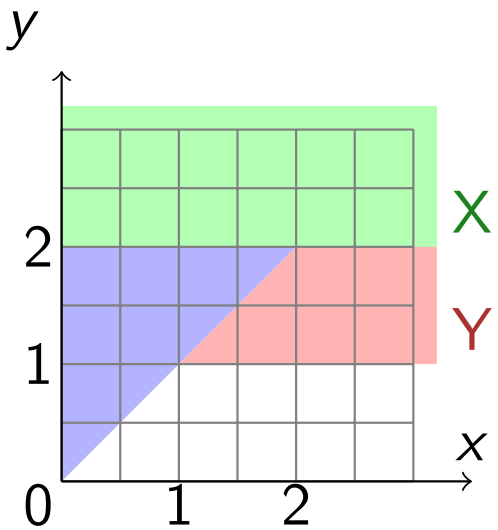

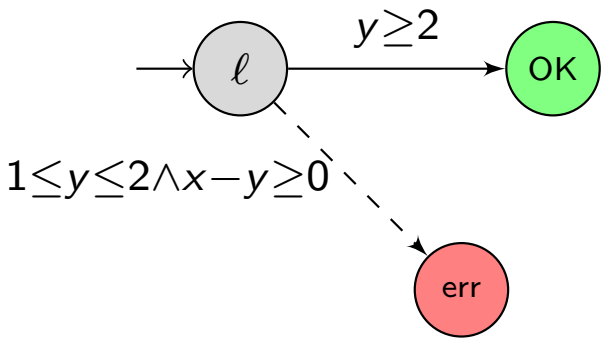

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Given $X, Y \subseteq \mathcal{Q}$, from which states can we delay into X while avoiding Y?

$$\mathrm{Pred}_{\geq 0}(X, Y) = \{q \in \mathcal{Q} \mid \exists d \geq 0, q + d \in X \wedge \forall d' \in [0, d], q + d' \notin Y\}.$$

Control and Environment Predecessors

Given $X \subseteq \mathcal{Q}$, let us define

$$\mathrm{Pred}_c(X) = \{q \in \mathcal{Q} \mid \exists q' \in X, \exists c, q \xrightarrow{c} q'\}, \quad (1)$$

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CPRE for Timed Games

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Define

$$\mathrm{CPRE}(X) = \mathrm{Pred}_{\geq 0}(\mathrm{Pred}_c(X), \mathrm{Pred}_u(\mathcal{Q} \setminus X)).$$

"The states from which one can delay into $\mathrm{Pred}_c(X)$ (and go to X), while avoiding environment's actions into $\mathcal{Q} \setminus X$."

CPRE for Timed Games

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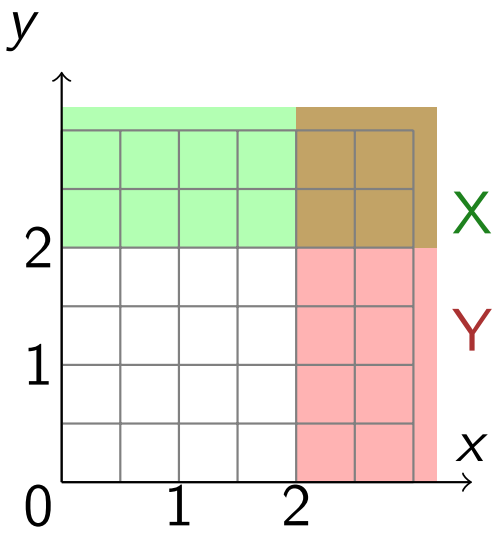

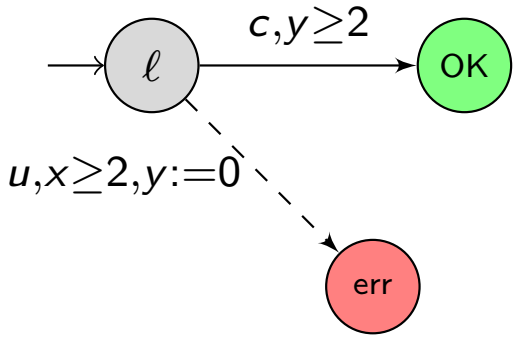

- $X = \mathit{OK} \times \mathbb{R}^C_{\geq 0}$,
- $\mathrm{Pred}_c(\mathrm{OK} \times \mathbb{R}^C_{\geq 0}) = (\ell, y \geq 2)$,
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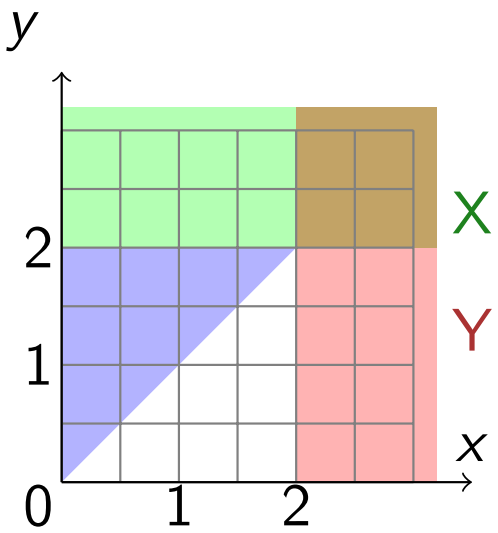

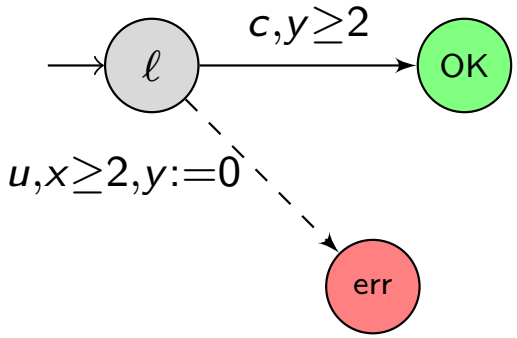

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CPRE for Timed Games

$$\mathrm{CPRE}(X) = \mathrm{Pred}_{\geq 0}(\mathrm{Pred}_c(X), \mathrm{Pred}_u(\mathcal{Q} \setminus X)).$$

Solving Timed Games

The following fixpoint is the set of states from which Controller can ensure reaching T: $\mu X.\big(T \cup X \cup \mathrm{CPRE}(X)\big)$.

In other terms, the algorithm is the following.

$X := T$
while $X \neq X \cup \mathrm{CPRE}(X)$ do
$X := X \cup \mathrm{CPRE}(X)$

Complexity

Determining if Controller has a winning strategy from a given state in a timed game is EXPTIME-complete.

CPRE for Timed Games

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Solving Timed Games

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How do we actually compute CPRE(X)?

Which data structure?

Which precise algorithm?

while $X \neq X \cup \mathrm{CPRE}(X)$ do

$X := X \cup \mathrm{CPRE}(X)$

Complexity

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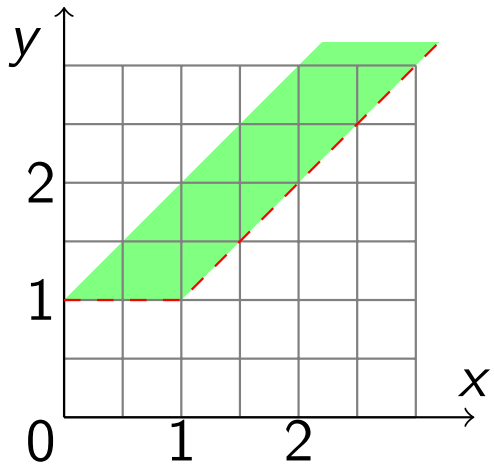
Outline

Difference-Bound Matrices (DBM)

Consider clocks x, y and the following set of states:

$$x < y \quad \wedge \quad y \leq x + 1 \quad \wedge \quad x \geq 0 \quad \wedge \quad y > 1$$



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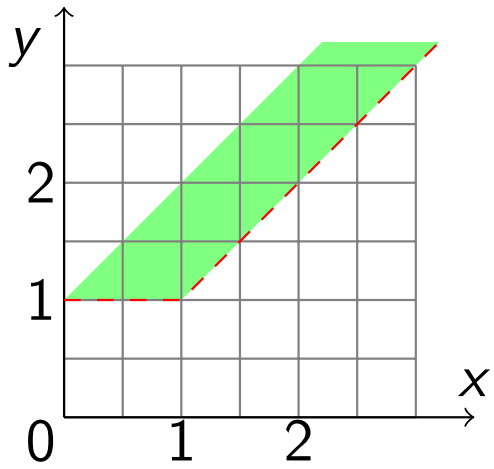
$$x - y < 0 \quad \wedge \quad y - x \leq 1 \quad \wedge \quad -x \leq 0 \quad \wedge \quad -y < -1$$



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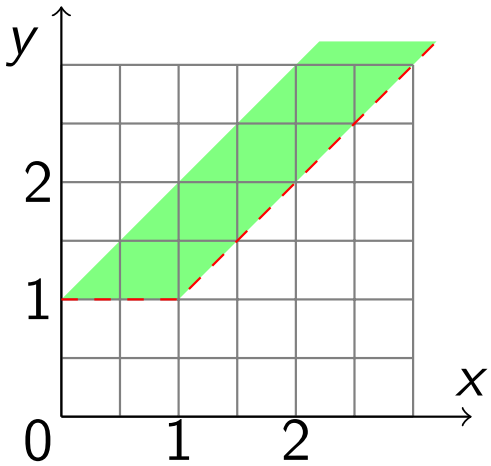

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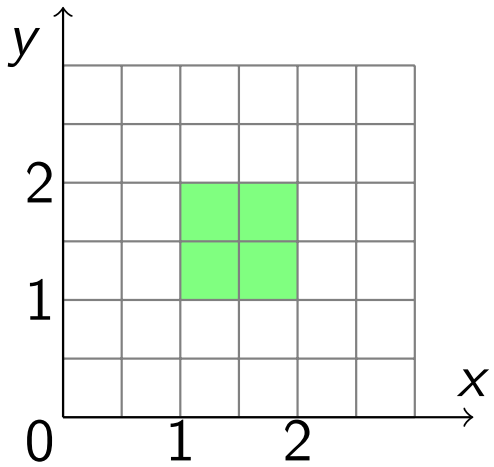
Difference-Bound Matrices (DBM)- Canonical Form

$$x \leq 2 \quad \wedge \quad -x \leq -1 \quad \wedge \quad y \leq 2 \quad \wedge \quad -y \leq -1$$



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DBM in canonical form: All constraints are tight.

Any given DBM can be made canonical in time $O(C^3)$.

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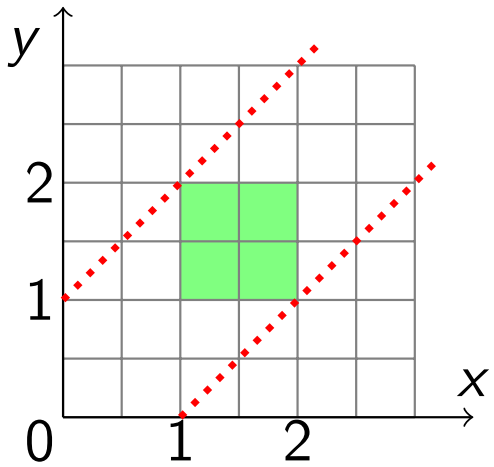

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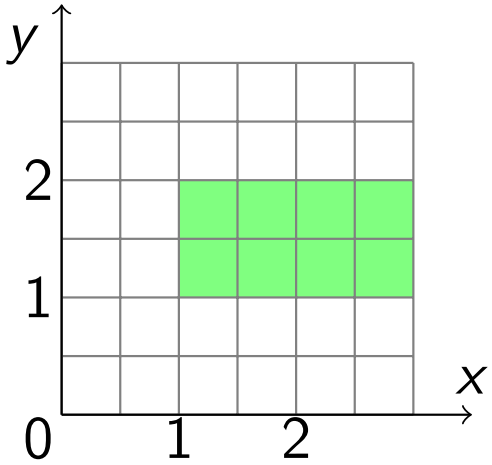
Difference-Bound Matrices (DBM) - Operations

Intersection



Difference-Bound Matrices (DBM) - Operations

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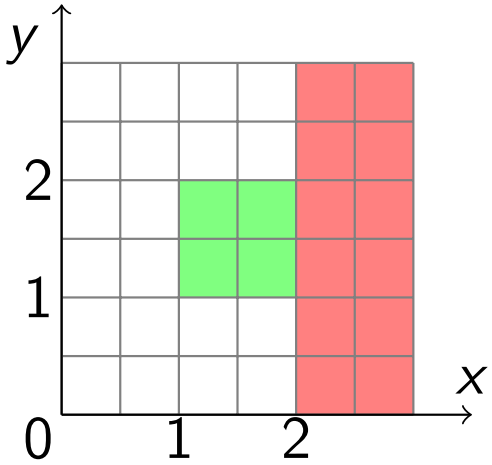

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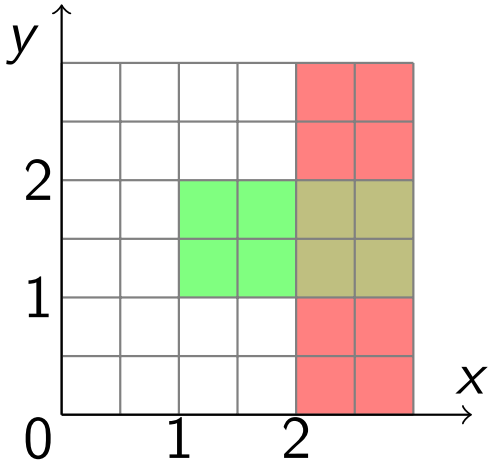


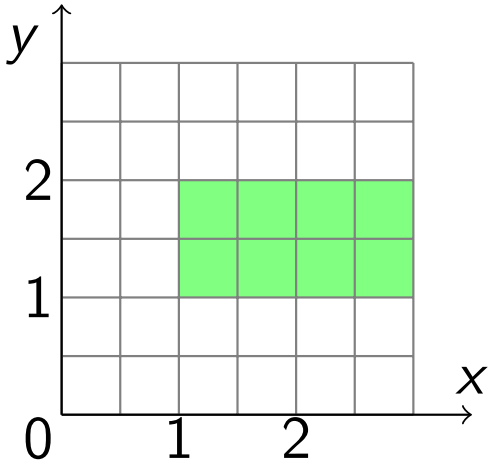
Idea: $y \leq 2$ **and** $y < 3 \;=\; y \leq 2$.

Given two DBMs M, N, $M \cap N$ is defined by

$$\forall i, j \in \{0, 1, \ldots, C\}, (M \cap N)_{i,j} = \min(M_{i,j}, N_{i,j}).$$

Difference-Bound Matrices (DBM) - Operations

Time Successors



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Difference-Bound Matrices (DBM) - Operations

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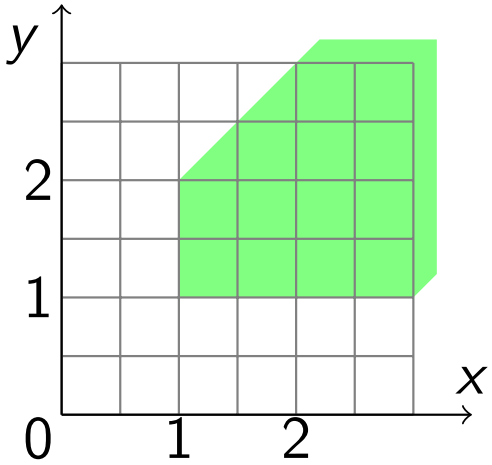

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Idea: Remove all upper bounds on clocks: $x \leq 3 \quad \rightarrow \quad x < \infty$.

Difference-Bound Matrices (DBM) - Operations

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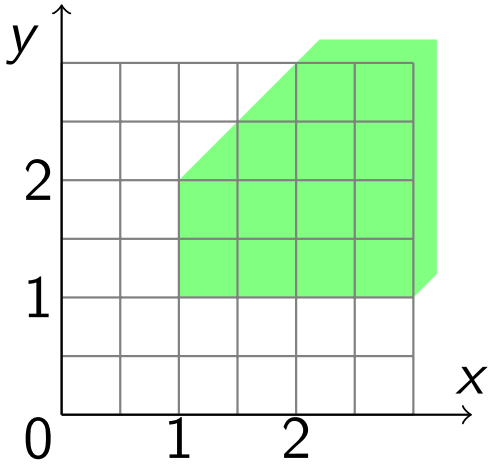

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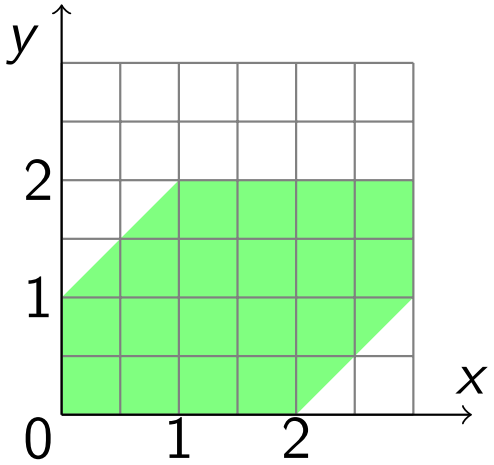

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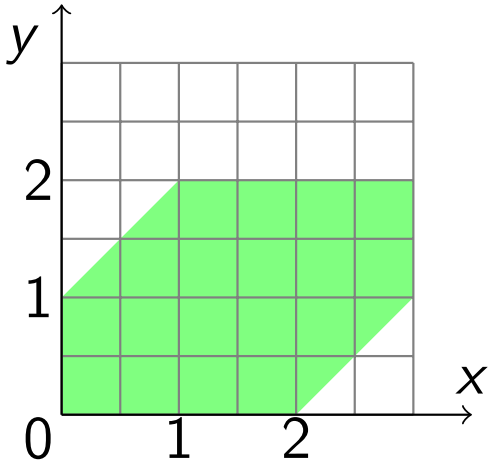
Idea: Remove all lower bounds on clocks: $x \geq 1 \quad \rightarrow \quad x \geq 0$.

Difference-Bound Matrices (DBM) - Operations

Time Predecessors



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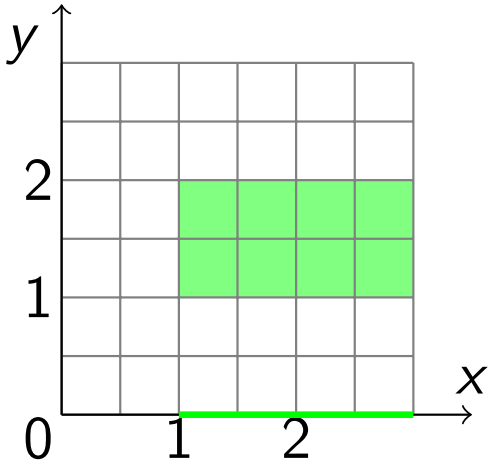
Clock Reset



Idea: Remove any constraint on reset clock y, and intersect with the DBM $y = 0$.

DBM Operations

- DBMs represent **convex** subsets of states
- Inclusion and Equality can be checked as well (not shown here)
- Successor and predecessors (through a given edge) can be computed efficiently $O(C^3)$

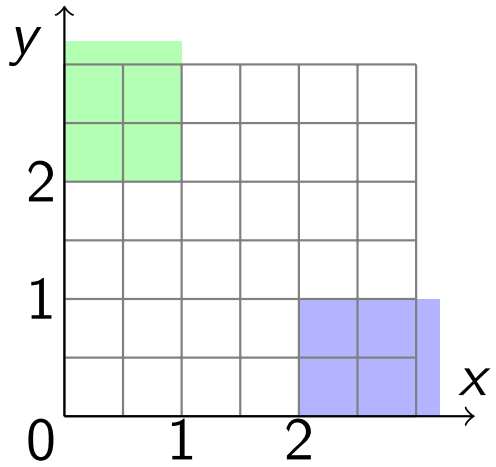
  e.g. for successor of set M through an edge with guard G, resetting x:

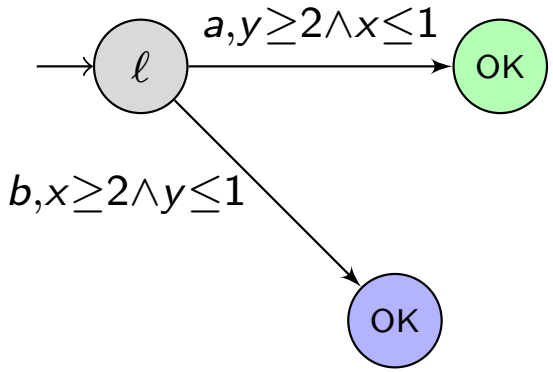
$$\mathrm{Post}_e(M) = \mathrm{Reset}_x\big(\mathrm{Post}_{\geq 0}(M) \cap G\big)$$

Can we compute CPRE with DBMs?

CPRE is not convex

Computing $\mathsf{CPRE}(\{\mathsf{OK}\}) = \mathsf{Pred}_{\geq 0}\big(\mathsf{Pred}_c(\{\mathsf{OK}\}), \emptyset\big)$.





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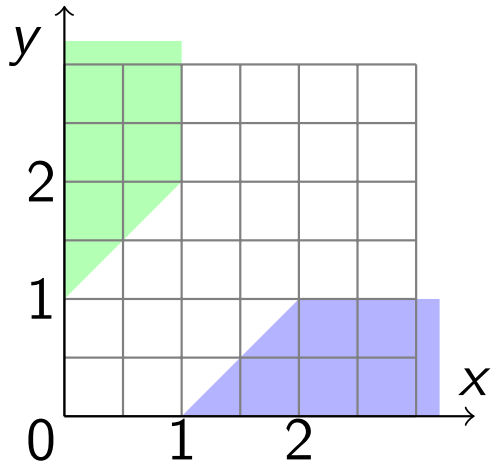

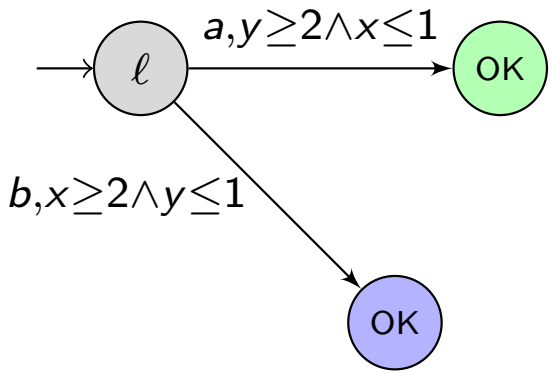

Non-convex...

Moreover we also need **complementation** which is also a non-convex operation

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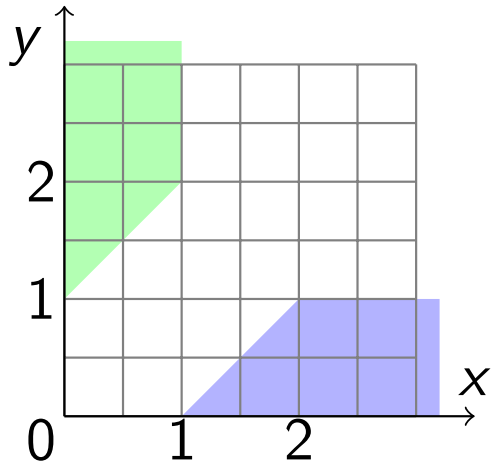

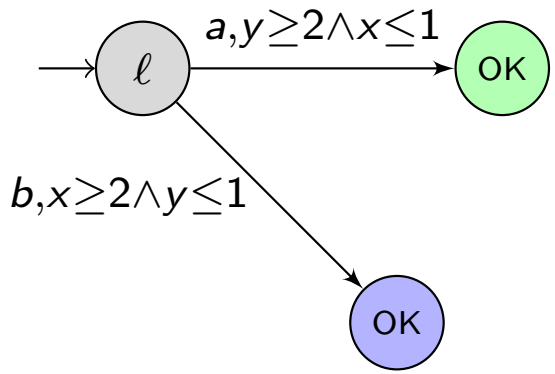

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DBM Federation

A list of DBMs which represent the **union** of each DBM

DBM Federations – Operations

Some easy operations:

- $(M_1 \cup \ldots \cup M_k) \cap (N_1 \cup \ldots \cup N_l) = \cup_{i,j} M_i \cap N_j$.

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To compute N^c,

- Write $N = C_1 \wedge C_2 \wedge \ldots \wedge C_m$, and $N^c = C_1^c \vee \ldots \vee C_m^c$.
- N^c is the federation: $C_1^c \cup \ldots \cup C_m^c$.

Example: $N = x \leq 2 \wedge y > 1 \quad \rightarrow \quad N^c = ((x < 2) \cup (y \leq 1))$.

DBM Federations – Operations

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Complementation and intersection are expensive...

▶ But there are a lot of heuristics to reduce redundant constraints, and minimize the size of federations...

Algorithm

Computation of the winning states using DBM federations:

$$
\begin{array}{l}
X := T \\
\text{while } X \neq X \cup \mathrm{CPRE}(X) \text{ do} \\
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$$

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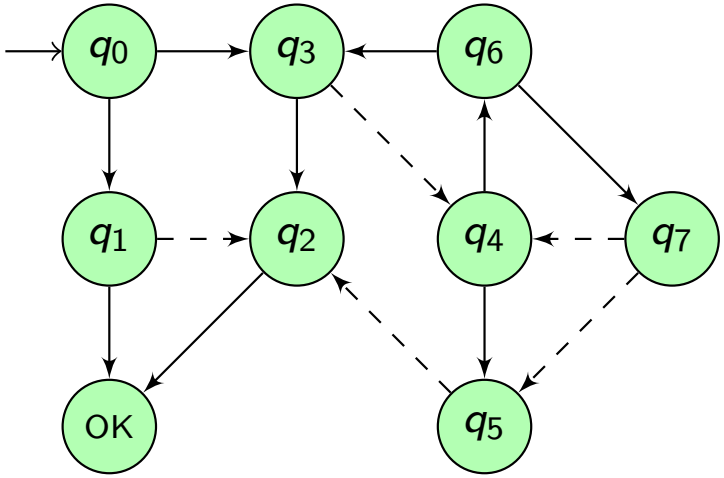
- In this computation, federations can sometimes become too big to handle... unions of thousands of DBMs in each federation
- This is a **backward** algorithm that computes **all** winning states

  If we are interested in **some** winning strategy, do we actually need to compute all winning states?

Outline

Smaller Solutions



- Backwards algorithm computes $X_1, \ldots, X_n$ such that

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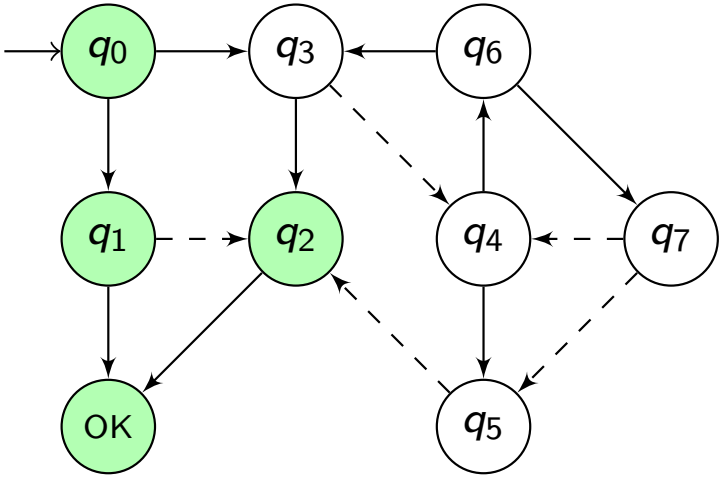
Smaller Solutions



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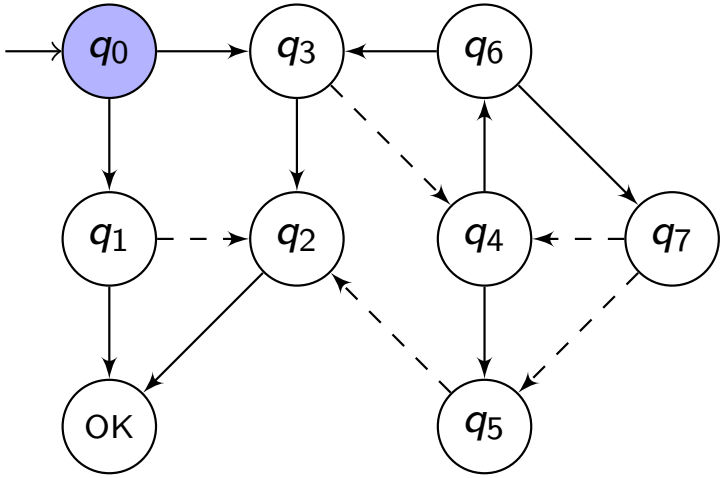
- A smaller solution could be given with $Y_1 = \{\mathsf{OK}\}$, Y_2, $Y_3, \ldots$, and

$$\forall i \geq 2, Y_i \subseteq \mathsf{CPRE}(Y_1 \cup \ldots \cup Y_{i-1}).$$

Forward Algorithm for Finite Games



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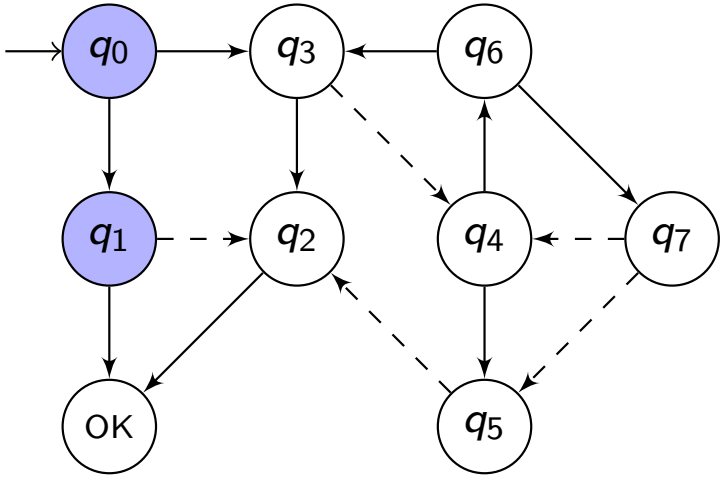

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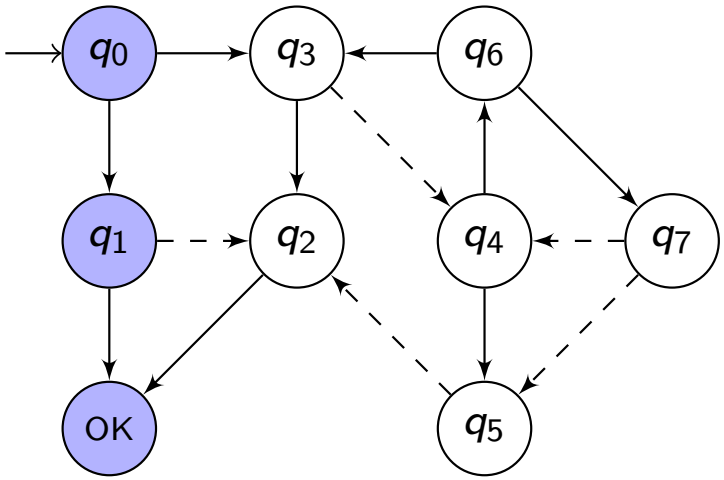

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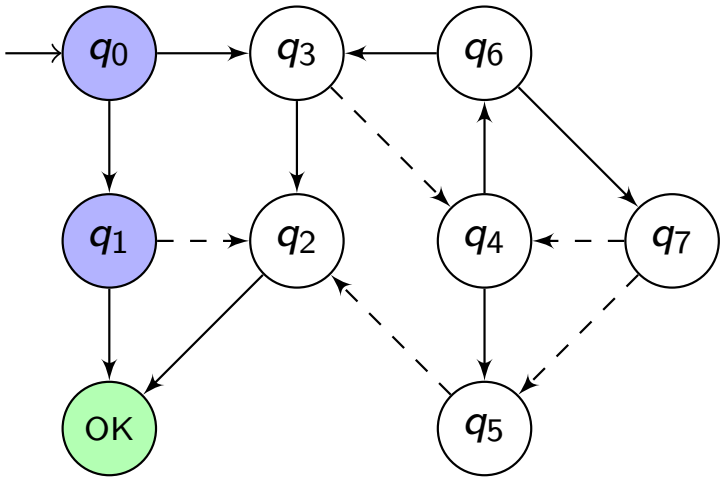

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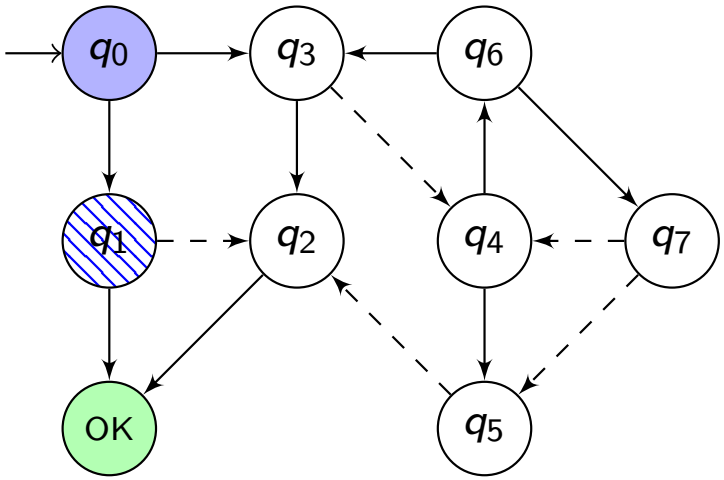

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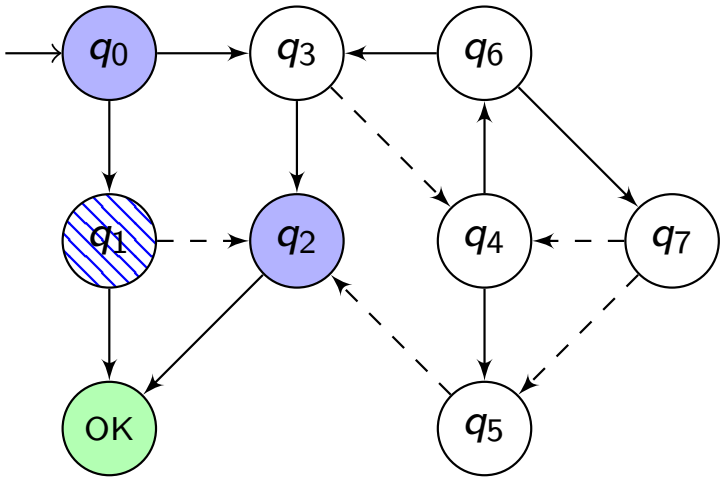

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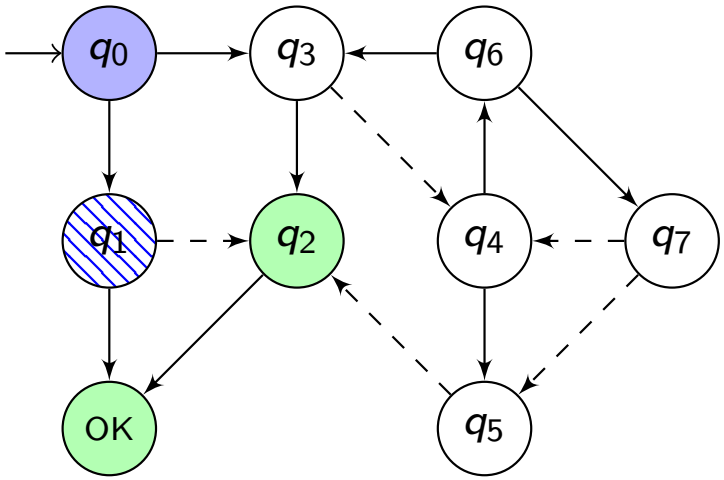

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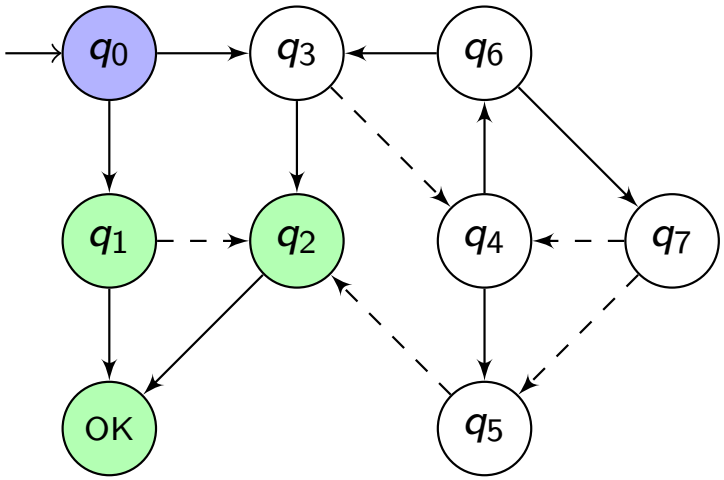

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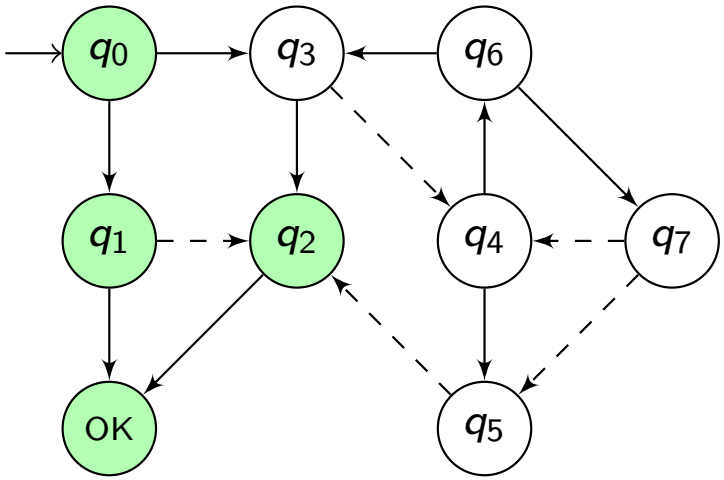

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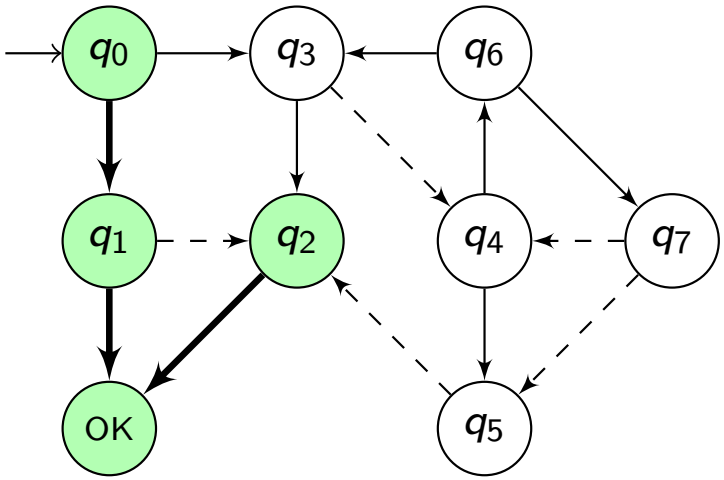

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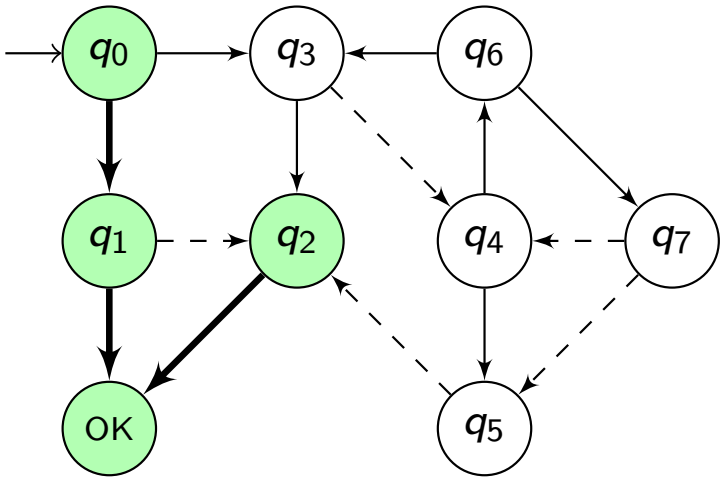

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Timed games: Apply same idea:

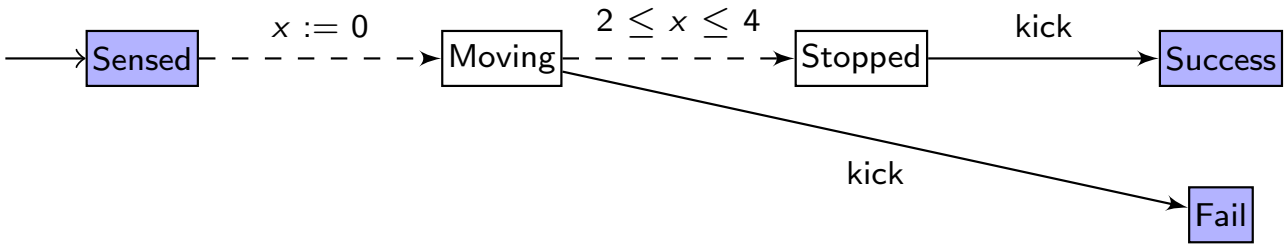
- Forward exploration using zones
- Whenever a target state is found, **backtrack:**
  - Explore all uncontrollable edges that haven't been explored
  - Use the CPRE operator to backtrack

Outline

Partial Observation

Previous algorithms assumed that the controller has **precise knowledge** about the system's state, at any moment.

Consider the following system in which the controller can only **sense** when the system enters a blue state:



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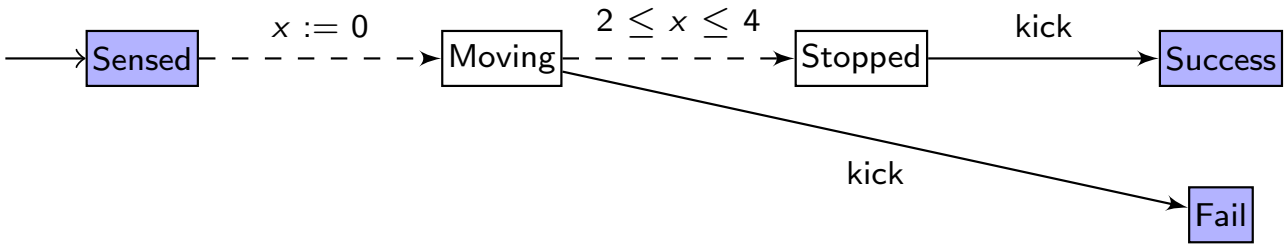

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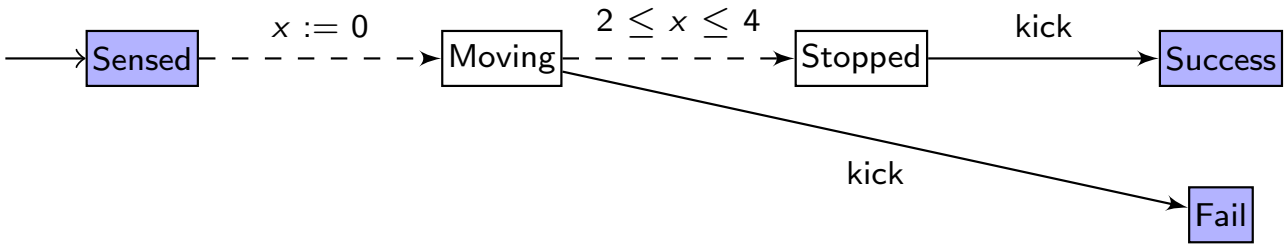

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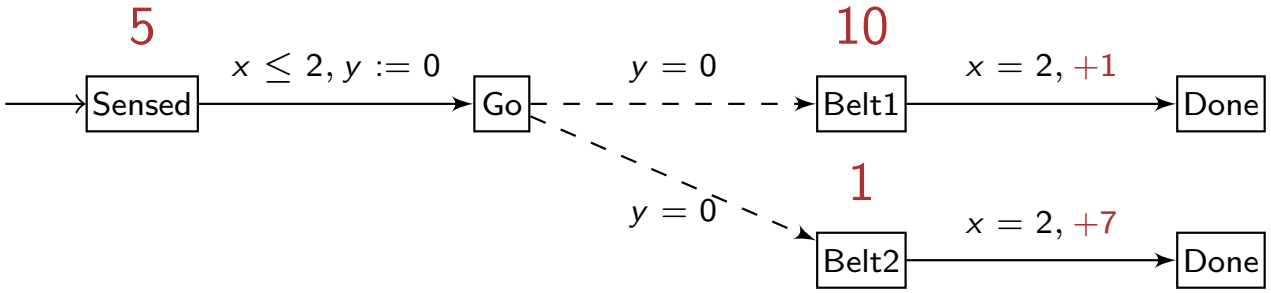
▶ Under some assumptions on the controller type, the problem is decidable, and the previous forward algorithm can be generalized.

Optimal Controllers

The previously presented algorithms compute an **arbitrary** controller.
Assume we want to compute the cheapest, or the fastest controller.



Weighted timed games: the cost of spending t time units at location ℓ is $C(\ell) \cdot t$, and each transition has a cost.

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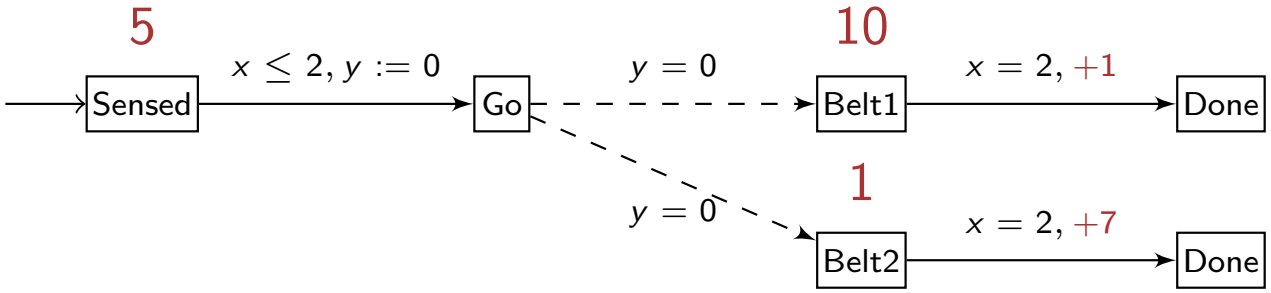

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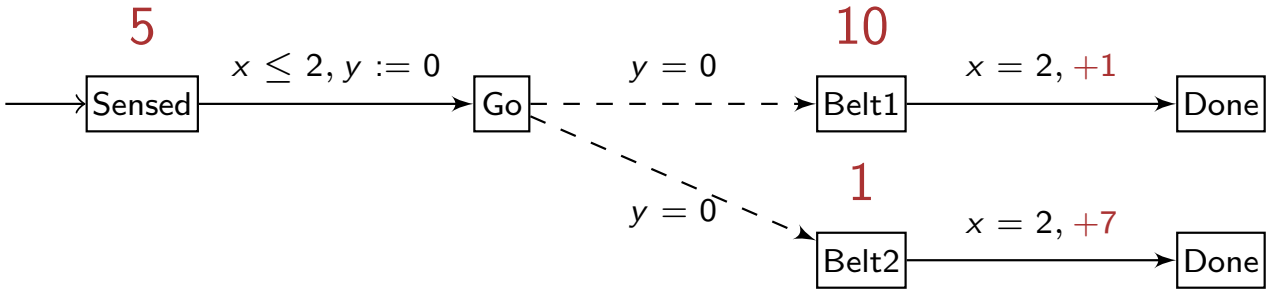

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Cost of the path through Belt1: $5t + 10(2 - t) + 1$
Cost of the path through Belt2: $5t + (2 - t) + 7$
The optimal cost:

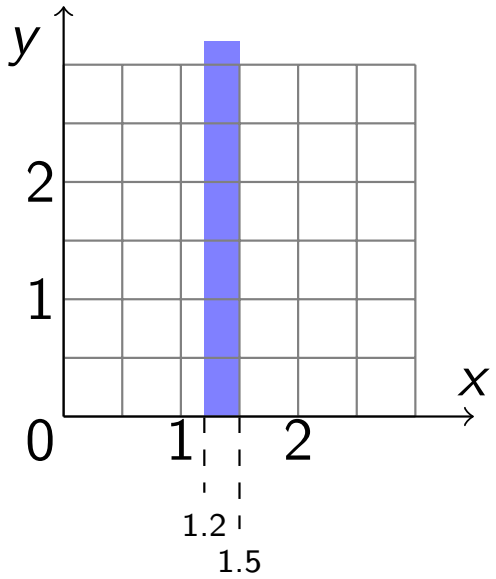
$$14 + \frac{1}{3} = \min_{t \in [0,2]} (\max(5t + 10(2 - t) + 1, 5t + (2 - t) + 7)).$$

Optimal Controllers - 2

$14 + \frac{1}{3} = \min_{t \in [0,2]}(\max(5t + 10(2-t) + 1, 5t + (2-t) + 7))$.

the min. is achieved for $t = \frac{4}{3}$.

Assume we want to reach the target with cost ≤ 15.



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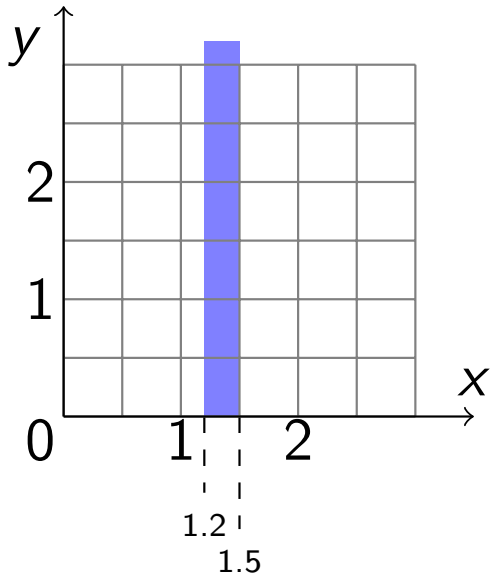

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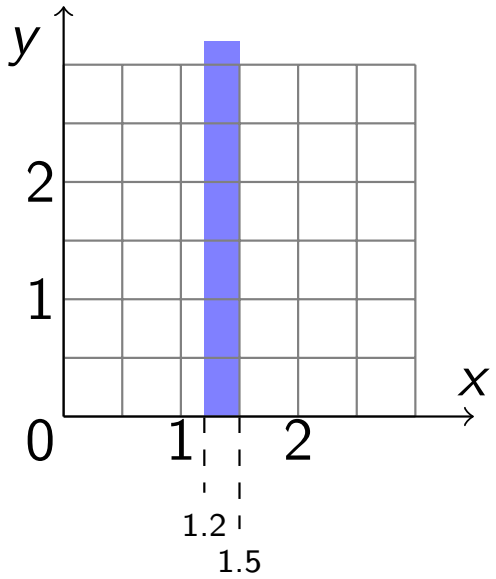

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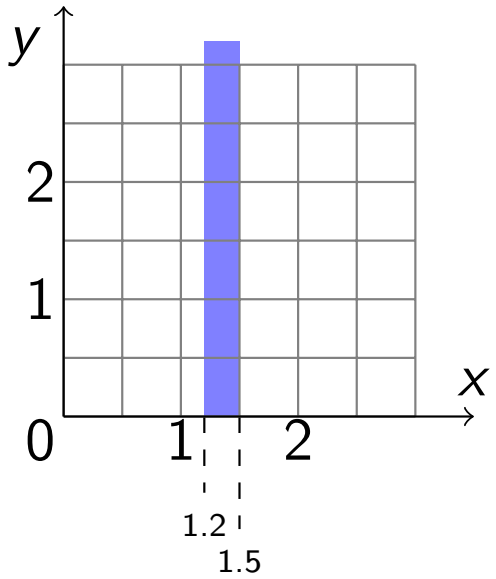

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→ Successor computation more complicated
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→ Undecidable but decidable for some classes with a single clock
→ Optimal cost can be approximated under some assumptions

Conclusion

Timed Games

- Formalism to model timed systems and controller synthesis problems.
- Backward algorithm: simple and computes **all** winning states
  Forward algorithm: more efficient in general to compute **some** winning strategy
- Presented algorithms are based on zones and are **semi-symbolic**: Discrete states are enumerated one by one, but sets of clock valuations are efficiently represented by zones
  - Good efficiency when small number of discrete states, but complex timing constraints (e.g. big constants)
  - Poor efficiency when a large number of discrete states
- Extensions: partial-observation, weights...
- Tool support: Uppaal-TIGA (see www.uppaal.org)

References

Timed automata and games

- Alur, Dill. A theory of timed automata. 1994.
- Asarin, O. Maler and A.Pnueli, Symbolic Controller Synthesis for Discrete and Timed Systems. 1995.
- E. Asarin, O. Maler, A.Pnueli, J. Sifakis, Controller Synthesis for Timed Automata. 1998.

Zone-based exploration

- Berthomieu, Menasch. An Enumerative Approach For Analyzing Time Petri Nets. 1983.
- Dill. Timing Assumptions and Verification of Finite-State Concurrent Systems. 1989.
- Bellman. Dynamic Programming. 1957. **(Difference-Bound Matrices)**
- Bengtsson, Yi. Timed Automata: Semantics, Algorithms and Tools. 2004. **(Survey)**

Controller Synthesis Algorithms using Zones

- Tripakis, Altisen. On-the-Fly Controller Synthesis for Discrete and Timed Systems. 1999.
- Cassez, David, Fleury, Larsen, Lime. Efficient On-the-Fly Algorithms for the Analysis of Timed Games. 2005.

Forward Algorithm

- Liu, Smolka. Simple Linear-Time Algorithms for Minimal Fixed Points (Extended Abstract). 1998.